ECS332 2019/1 Part II.3 Dr.Prapun

4.4 Classical DSB-SC Modulators

To produce the modulated signal $A_c \cos(2\pi f_c t)m(t)$, we may use the following methods which generate the modulated signal along with other signals which can be eliminated by a bandpass filter restricting frequency contents to around f_c.

4.55. Multiplier Modulators [6, p 184] or **Product Modulator**[3, p 180]: Here modulation is achieved directly by multiplying $m(t)$ by $\cos(2\pi f_c t)$ using an analog multiplier whose output is proportional to the product of two input signals.

- Such a multiplier may be obtained from
  - (a) a variable-gain amplifier in which the gain parameter (such as the the β of a transistor) is controlled by one of the signals, say, $m(t)$. When the signal $\cos(2\pi f_c t)$ is applied at the input of this amplifier, the output is then proportional to $m(t)\cos(2\pi f_c t)$.
  - (b) two logarithmic and an antilogarithmic amplifiers with outputs proportional to the log and antilog of their inputs, respectively.
    - Key equation:
$$A \times B = e^{(\ln A + \ln B)}.$$

4.56. When it is easier to build a squarer than a multiplier, we may use a **square modulator** shown in Figure 25.

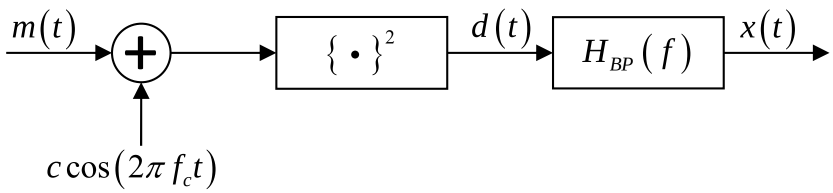
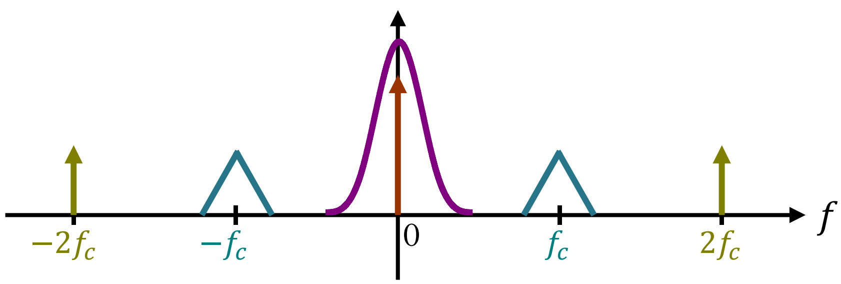


Figure 25: Block diagram of a square modulator

Note that

$$\begin{aligned} d(t) &= (m(t) + c\cos(2\pi f_c t))^2 \\ &= m^2(t) + 2cm(t)\cos(2\pi f_c t) + c^2\cos^2(2\pi f_c t) \\ &= m^2(t) + 2cm(t)\cos(2\pi f_c t) + \frac{c^2}{2} + \frac{c^2}{2}\cos(2\pi(2f_c)t) \end{aligned}$$



Using a band-pass filter (BPF) whose frequency response is

$$H_{BP}(f) = \begin{cases} g, & |f - f_c| \le B, \\ g, & |f - (-f_c)| \le B, \\ 0, & \text{otherwise}, \end{cases} \tag{59}$$

we can produce $2cgm(t)\cos(2\pi f_c t)$ at the output of the BPF. In particular, choosing the gain g to be $(c\sqrt{2})^{-1}$, we get $m(t) \times \sqrt{2}\cos(2\pi f_c t)$.

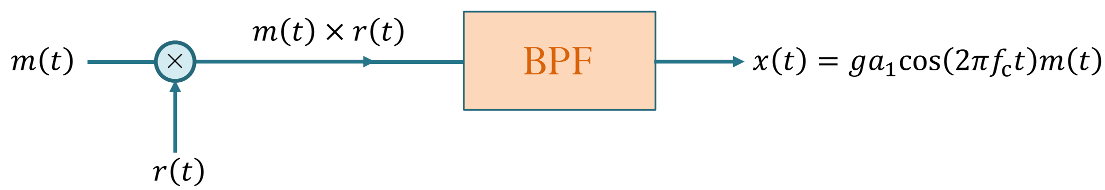
- Alternative, can use $\left(m(t) + c\cos\left(\frac{\omega_c}{2}t\right)\right)^3$.

4.57. Another conceptually nice way to produce a signal of the form $A_c m(t)\cos(2\pi f_c t)$ is to

(1) multiply $m(t)$ by "any" **periodic and even** signal $r(t)$ whose period is $T_c = \frac{1}{f_c}$

and then

(2) pass the result though a BPF used in (59).

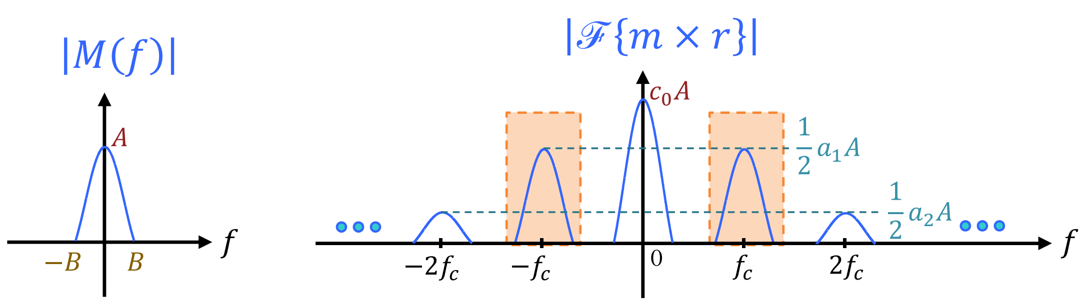


To see how this works, recall that because $r(t)$ is an even function, we know that

$$r(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos\left(2\pi(kf_c)t\right) \text{ for some } c_0, a_1, a_2, \ldots.$$

Therefore,

$$m(t)r(t) = c_0 m(t) + \sum_{k=1}^{\infty} a_k m(t)\cos\left(2\pi(kf_c)t\right).$$

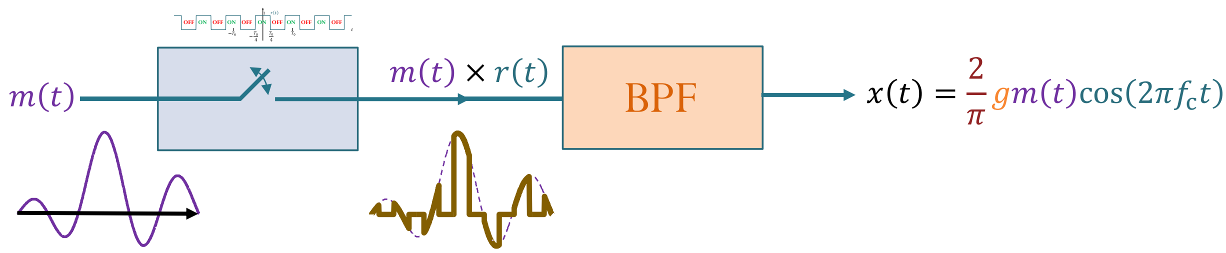


See also [5, p 157]. In general, for this scheme to work, we need

- $a_1 \neq 0$ period of r;
- $f_c > 2B$ (to prevent overlapping).

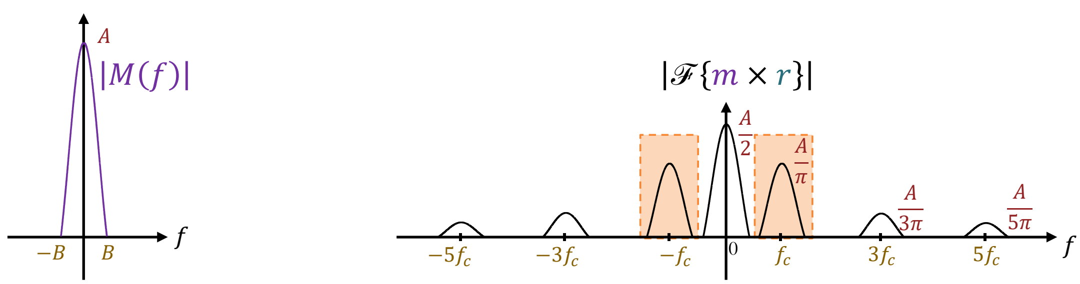
Note that if $r(t)$ is not even, then by (50c), the resulting modulated signal will have the form $x(t) = a_1 m(t)\cos(2\pi f_c t + \phi_1)$.

4.58. ***Switching modulator***: An important example of a periodic and even function $r(t)$ is the square pulse train considered in Example 4.48. Recall that multiplying this $r(t)$ to a signal $m(t)$ is equivalent to switching $m(t)$ on and off periodically.

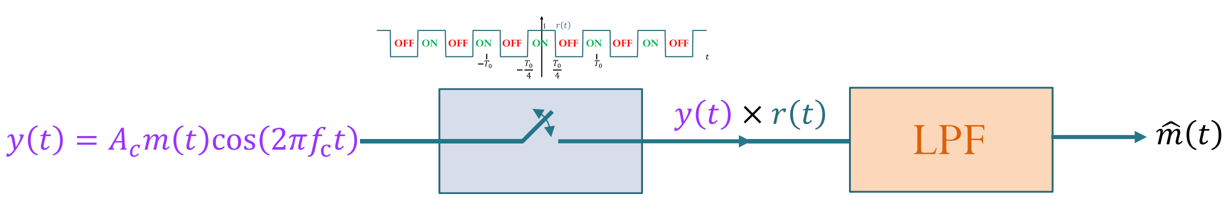


$$r(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi f_c t) - \frac{2}{3\pi}\cos(2\pi(3f_c)t) + \frac{2}{5\pi}\cos(2\pi(5f_c)t) + \cdots$$

$$m(t) \times r(t) = \frac{1}{2}m(t) + \frac{2}{\pi}m(t)\cos(2\pi f_c t) - \frac{2}{3\pi}m(t)\cos(2\pi(3f_c)t) + \frac{2}{5\pi}m(t)\cos(2\pi(5f_c)t) + \cdots$$



4.59. ***Switching Demodulator***: The switching technique can also be used at the demodulator as well.



We have seen that, for DSB-SC modem, the key equation is given by (41). When switching demodulator is used, the key equation is

$$\mathrm{LPF}\{m(t)\cos(2\pi f_c t) \times 1[\cos(2\pi f_c t) \geq 0]\} = \frac{1}{\pi}m(t) \qquad (60)$$

[5, p 162].

$$r(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi f_c t) - \frac{2}{3\pi}\cos(2\pi(3f_c)t) + \frac{2}{5\pi}\cos(2\pi(5f_c)t) + \cdots$$

$$y(t)r(t) = \frac{1}{2}y(t) + \frac{2}{\pi}y(t)\cos(2\pi f_c t) - \frac{2}{3\pi}y(t)\cos(2\pi(3f_c)t) + \frac{2}{5\pi}y(t)\cos(2\pi(5f_c)t) + \cdots$$

$$\begin{aligned}
&= \frac{1}{2}A_c m(t)\cos(2\pi f_c t) \\
&\quad + \frac{2}{\pi}A_c m(t)\cos(2\pi f_c t)\cos(2\pi f_c t) \\
&\quad - \frac{2}{3\pi}A_c m(t)\cos(2\pi f_c t)\cos(2\pi(3f_c)t) \\
&\quad + \frac{2}{5\pi}A_c m(t)\cos(2\pi f_c t)\cos(2\pi(5f_c)t) + \cdots
\end{aligned}$$

$$\cos(A)\cos(B) = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$$

$$\begin{aligned}
&= \frac{1}{2}A_c m(t)\cos(2\pi f_c t) \\
&\quad + \frac{1}{\pi}A_c m(t)\left(1 + \cos(2\pi(2f_c)t)\right) \\
&\quad - \frac{1}{3\pi}A_c m(t)\left(\cos(2\pi(2f_c)t) + \cos(2\pi(4f_c)t)\right) \\
&\quad + \frac{1}{5\pi}A_c m(t)\left(\cos(2\pi(4f_c)t) + \cos(2\pi(6f_c)t)\right) + \cdots
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}A_c m(t)\cos(2\pi f_c t) \\
&\quad \boxed{+\frac{1}{\pi}A_c m(t)} + \frac{1}{\pi}A_c m(t)\cos(2\pi(2f_c)t) \\
&\quad - \frac{1}{3\pi}A_c m(t)\cos(2\pi(2f_c)t) - \frac{1}{3\pi}A_c m(t)\cos(2\pi(4f_c)t) \\
&\quad + \frac{1}{5\pi}A_c m(t)\cos(2\pi(4f_c)t) + \frac{1}{5\pi}A_c m(t)\cos(2\pi(6f_c)t) + \cdots
\end{aligned}$$

Note that this technique still requires the switching to be in sync with the incoming cosine as in the basic DSB-SC.

4.5 (Standard) Amplitude modulation: AM

4.60. DSB-SC amplitude modulation (which is summarized in Figure 26) is easy to understand and analyze in both time and frequency domains. However, analytical simplicity is not always accompanied by an equivalent simplicity in practical implementation.

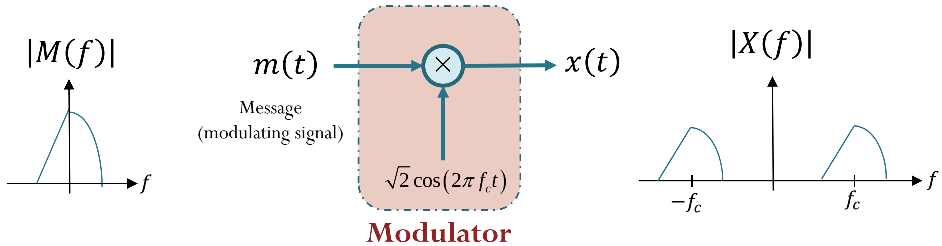


Figure 26: DSB-SC modulation.

Problem: The (coherent) demodulation of DSB-SC signal requires the receiver to possess a carrier signal that is synchronized with the incoming carrier. This requirement is not easy to achieve in practice because the modulated signal may have traveled hundreds of miles and could even suffer from some unknown frequency shift.

4.61. If a carrier component is transmitted along with the DSB signal, demodulation can be simplified.

(a) The received carrier component can be extracted using a narrowband bandpass filter and can be used as the demodulation carrier. (There is no need to generate a carrier at the receiver.)

(b) If the carrier amplitude is sufficiently large, the need for generating a demodulation carrier can be completely avoided.

- This will be the focus of this section.

Definition 4.62. For AM, the transmitted signal is typically defined as

$$x_{\mathrm{AM}}(t) = (A + m(t))\cos(2\pi f_c t) = \underbrace{A\cos(2\pi f_c t)}_{\text{carrier}} + \underbrace{m(t)\cos(2\pi f_c t)}_{\text{sidebands}}$$

Assumptions for $m(t)$:

(a) Band-limited to B; that is, $|M(f)| = 0$ for $|f| > B$.

(b) Bounded between $-m_p$ and m_p; that is, $|m(t)| \le m_p$.

4.63. Spectrum of $x_{\mathrm{AM}}(t)$:

- Basically the same as that of DSB-SC signal except for the two additional impulses (**discrete** spectral component) at the carrier frequency $\pm f_c$.
  - This is why we say the DSB-SC system is a ***suppressed carrier*** system.

Definition 4.64. Consider a signal $A(t)\cos(2\pi f_c t)$. If $A(t)$ varies slowly in comparison with the sinusoidal carrier $\cos(2\pi f_c t)$, then the ***envelope*** $E(t)$ of $A(t)\cos(2\pi f_c t)$ is $|A(t)|$.

4.65. ***Envelope of AM signal***: For AM signal, $A(t) \equiv A + m(t)$ and

$$E(t) = |A + m(t)|\,.$$

See Figure 27.

Case (a) If $\forall t, A(t) > 0$, then $E(t) = A(t) = A + m(t)$

- The envelope has the same shape as $m(t)$.
- Enable envelope detection: Extract $m(t)$ from the envelope.

Case (b) If $\exists t, A(t) < 0$, then $E(t) \neq A(t)$.

- The envelope shape differs from the shape of $m(t)$ because the negative part of $A + m(t)$ is rectified.
  - This is referred to as **phase reversal** and envelope distortion.

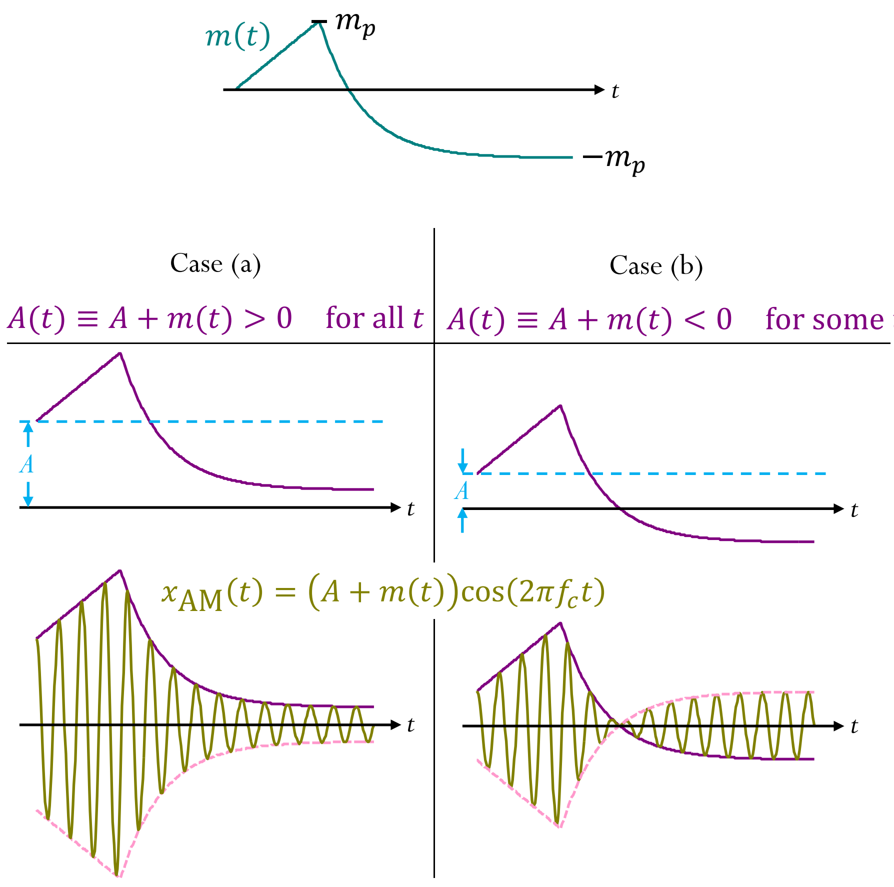


Figure 27: AM signal and its envelope [6, Fig 4.8]

Definition 4.66. The positive constant

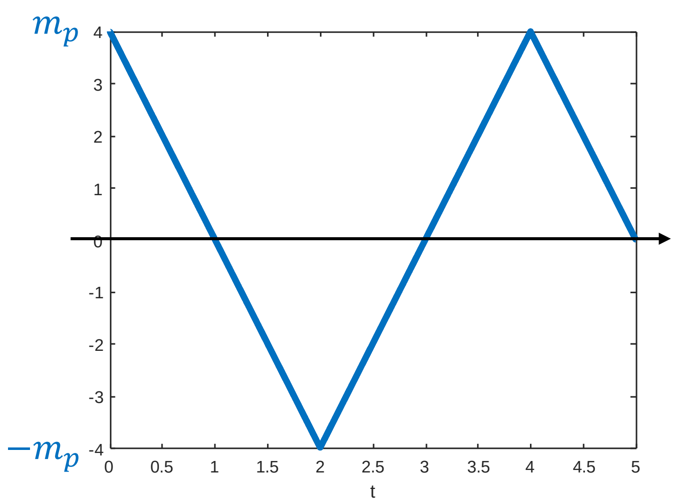
$$\mu \equiv \frac{\max_t (\text{envelope of the sidebands})}{\max_t (\text{envelope of the carrier})} = \frac{\max_t |m(t)|}{\max_t |A|} = \frac{m_p}{A}$$

is called the **modulation index**.

- The quantity $\mu \times 100\%$ is often referred to as the **percent modulation**.

Example: Modulation Index μ

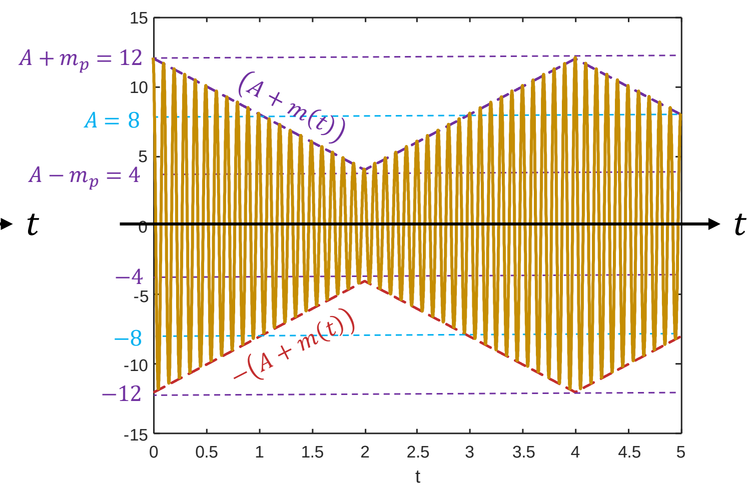
$m(t)$



$m_p = 4$

$\mu = 50\%$

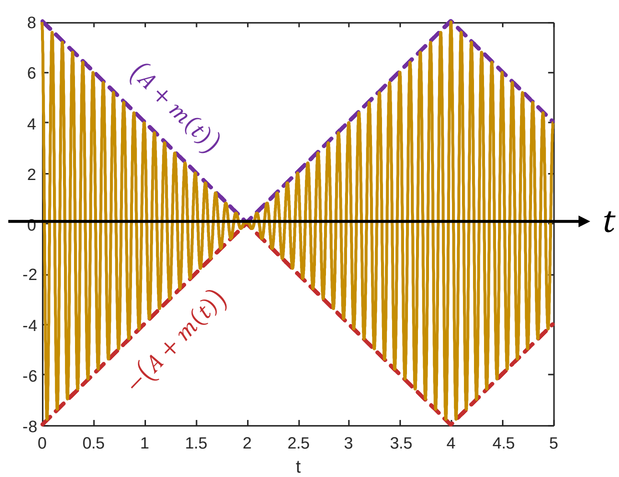
$x(t) = \big(A + m(t)\big)\cos(2\pi f_c t)$



$$\mu = \frac{m_p}{A} \Rightarrow A = \frac{m_p}{\mu} = \frac{4}{0.5} = 8$$

$\mu = 100\%$

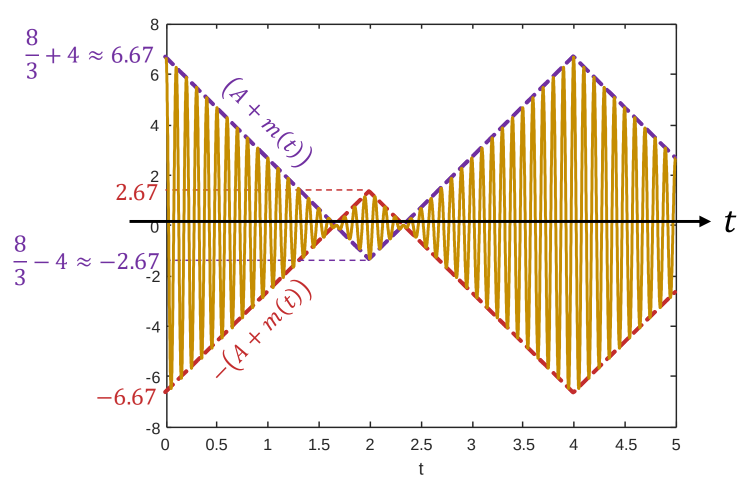
$x(t) = \big(A + m(t)\big)\cos(2\pi f_c t)$



$$\mu = \frac{m_p}{A} \Rightarrow A = \frac{m_p}{\mu} = \frac{4}{1} = 4$$

$\mu = 150\%$

$x(t) = \big(A + m(t)\big)\cos(2\pi f_c t)$



$$\mu = \frac{m_p}{A} \Rightarrow A = \frac{m_p}{\mu} = \frac{4}{1.5} = \frac{8}{3}$$

Example 4.67. Consider a sinusoidal (pure-tone) message $m(t) = A_m \cos(2\pi f_m t)$. Suppose $A = 1$. Then, $\mu = A_m$. Figure 28 shows the effect of changing the modulation index on the modulated signal.

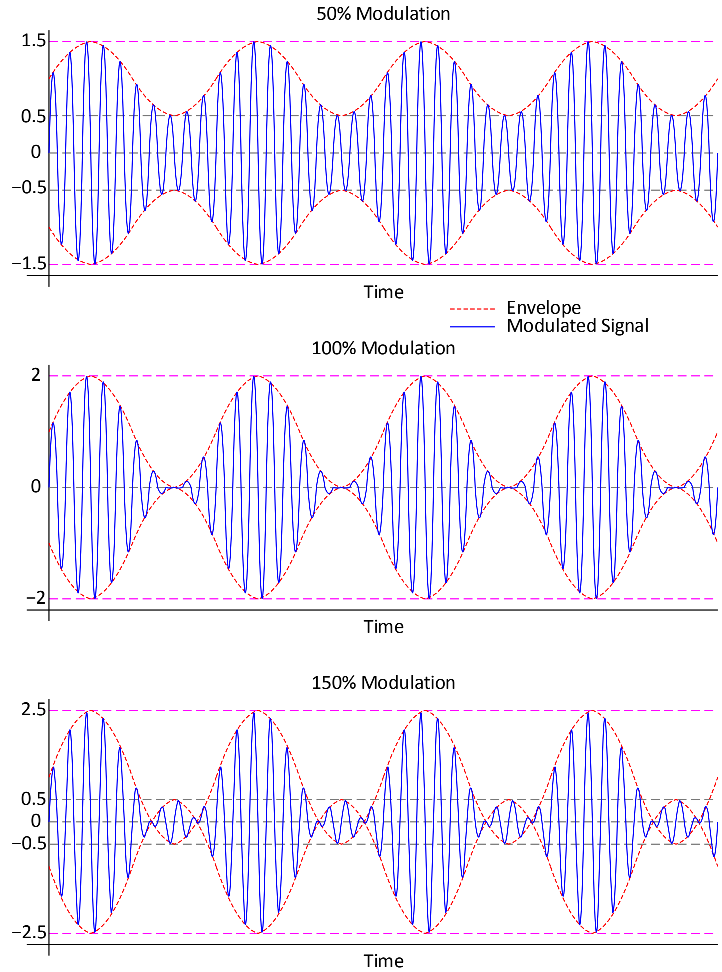


Figure 28: Modulated signal in standard AM with sinusoidal message

4.68. It should be noted that the ratio that defines the modulation index compares the maximum of the two envelopes. In other references, the notation for the AM signal may be different but the idea (and the corresponding motivation) that defines the modulation index remains the same.

- In [3, p 163], it is assumed that $m(t)$ is already scaled or normalized to have a magnitude not exceeding unity ($|m(t)| \leq 1$) [3, p 163]. There,

$$x_{\text{AM}}(t) = A_c(1 + \mu m(t))\cos(2\pi f_c t) = \underbrace{A_c \cos(2\pi f_c t)}_{\text{carrier}} + \underbrace{A_c \mu m(t) \cos(2\pi f_c t)}_{\text{sidebands}}.$$

- $m_p = 1$
- The modulation index is then

$$\frac{\max_t (\text{envelope of the sidebands})}{\max_t (\text{envelope of the carrier})} = \frac{\max_t |A_c \mu m(t)|}{\max_t |A_c|} = \frac{|A_c \mu|}{|A_c|} = \mu.$$

- In [15, p 116],

$$x_{\text{AM}}(t) = A_c \left(1 + \mu \frac{m(t)}{m_p}\right) \cos(2\pi f_c t) = \underbrace{A_c \cos(2\pi f_c t)}_{\text{carrier}} + \underbrace{A_c \mu \frac{m(t)}{m_p} \cos(2\pi f_c t)}_{\text{sidebands}}.$$

   - The modulation index is then

$$\frac{\max_t (\text{envelope of the sidebands})}{\max_t (\text{envelope of the carrier})} = \frac{\max_t \left|A_c \mu \frac{m(t)}{m_p}\right|}{\max_t |A_c|} = \frac{|A_c| \mu \frac{m_p}{m_p}}{|A_c|} = \mu.$$

4.69. Power of the transmitted signals.

(a) In DSB-SC system, recall, from 4.40, that, when

$$x(t) = m(t) \cos(2\pi f_c t)$$

with f_c sufficiently large, we have

$$P_x = \frac{1}{2} P_m.$$

Therefore, all transmitted power are in the sidebands which contain message information.

(b) In AM system,

$$x_{\text{AM}}(t) = \underbrace{A \cos(2\pi f_c t)}_{\text{carrier}} + \underbrace{m(t) \cos(2\pi f_c t)}_{\text{sidebands}}.$$

If we assume that the average of $m(t)$ is 0 (no DC component), then the spectrum of the sidebands $m(t)\cos(2\pi f_c t + \theta)$ and the carrier $A\cos(2\pi f_c t + \theta)$ are non-overlapping in the frequency domain. Hence, when f_c is sufficiently large

$$P_x = \frac{1}{2} A^2 + \frac{1}{2} P_m.$$

- Efficiency:

- For high power efficiency, we want small $\frac{m_p^2}{\mu^2 P_m}$.
  - By definition, $|m(t)| \le m_p$. Therefore, $\frac{m_p^2}{P_m} \ge 1$.

  - Want μ to be large. However, when $\mu > 1$, we have phase reversal. So, the largest value of μ is 1.
  - The best power efficiency we can achieved is then 50%.

- Conclusion: at least 50% (and often close to 2/3[3, p. 176]) of the total transmitted power resides in the carrier part which is independent of $m(t)$ and thus conveys no message information.

4.70. An AM signal can be demodulated using the same coherent demodulation technique that was used for DSB. However, the use of coherent demodulation negates the advantage of AM.

- Note that, conceptually, the received AM signal is the same as DSB-SC signal except that the $m(t)$ in the DSB-SC signal is replaced by $A(t) = A + m(t)$. We also assume that A is large enough so that $A(t) \ge 0$.

- Recall the key equation of ***switching demodulator*** (60):

$$\text{LPF}\{A(t)\cos(2\pi f_c t) \times 1[\cos(2\pi f_c t) \ge 0]\} = \frac{1}{\pi} A(t) \qquad (61)$$

  We noted before that this technique requires the switching to be in sync with the incoming cosine.

4.71. ***Demodulation*** of AM Signals via **rectifier detector**: The receiver will first recover $A + m(t)$ and then remove A.

- When $\forall t, A(t) \geq 0$, we can replace the switching demodulator by the ***rectifier demodulator/detector***. In which case, we suppress the negative part of $y(t) = x_{\mathrm{AM}}(t)$ using a diode (half-wave rectifier: HWR).
  - Here, we define a HWR to be a memoryless device whose input-output relationship is described by a function $f_{\mathrm{HWR}}(\cdot)$:

$$f_{\mathrm{HWR}}(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

- Surprisingly, this is mathematically equivalent to a switching demodulator in (60) and (61).

- It is in effect synchronous detection performed without using a local carrier [5, p 167].
- This method needs $A(t) \geq 0$ so that the sign of $A(t)\cos(2\pi f_c t)$ will be the same as the sign of $\cos(2\pi f_c t)$.
- The dc term $\frac{A}{\pi}$ may be blocked by a capacitor to give the desired output $m(t)/\pi$.

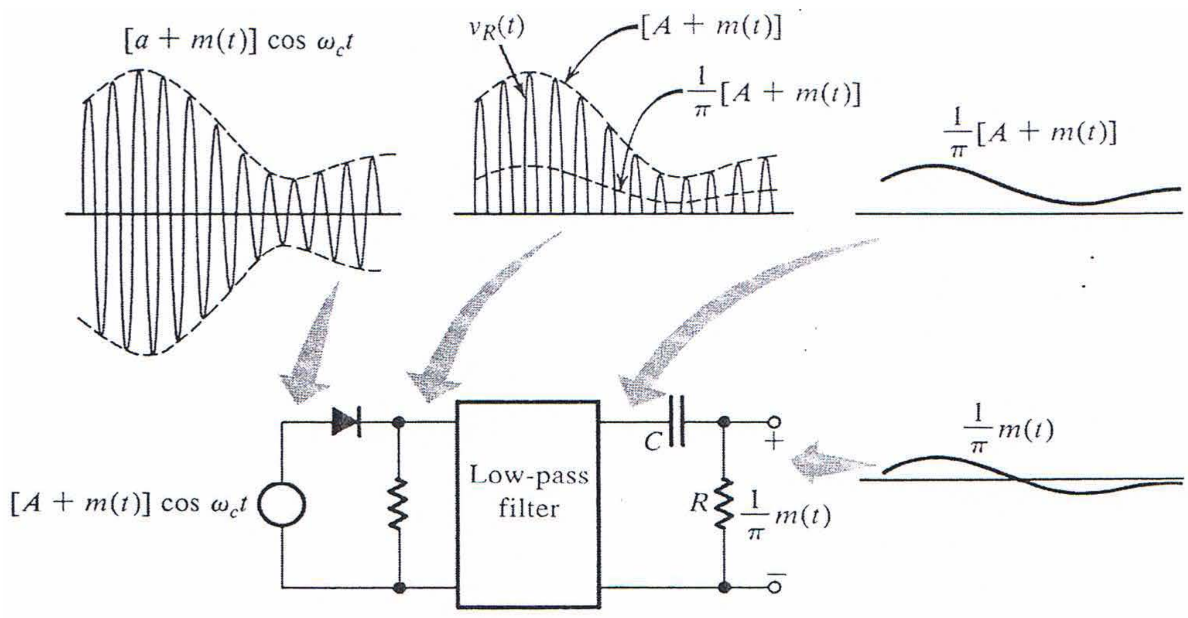


Figure 29: Rectifier detector for AM [6, Fig. 4.10].

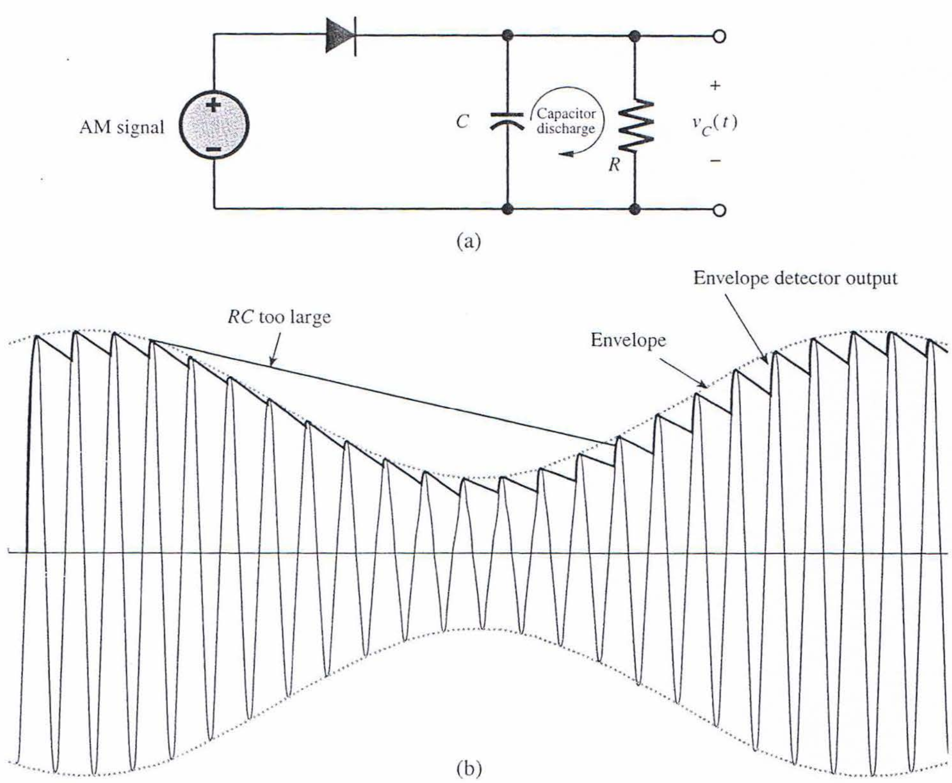


Figure 30: Envelope detector for AM [6, Fig. 4.11].

4.72. Demodulation of AM signal via ***envelope detector***:

- Design criterion of RC:

$$2\pi B \ll \frac{1}{RC} \ll 2\pi f_c.$$

- The envelope detector output is $A + m(t)$ with a ripple of frequency f_c.
- The dc term can be blocked out by a capacitor or a simple RC high-pass filter.
- The ripple may be reduced further by another (low-pass) RC filter.

4.73. AM Trade-offs:

(a) *Disadvantages*:

- Higher power and hence higher cost required at the transmitter
- The carrier component is wasted power as far as information transfer is concerned.
- Bad for power-limited applications.

(b) *Advantages*:

- Coherent reference is not needed for demodulation.
- Demodulator (receiver) becomes simple and inexpensive.
- For broadcast system such as commercial radio (with a huge number of receivers for each transmitter),
  - any cost saving at the receiver is multiplied by the number of receiver units.
  - it is more economical to have one expensive high-power transmitter and simpler, less expensive receivers.

(c) Conclusion: Broadcasting systems tend to favor the trade-off by migrating cost from the (many) receivers to the (fewer) transmitters.

4.74. References: [3, p 198–199], [6, Section 4.3] and [14, Section 3.1.2].

4.6 Bandwidth-Efficient Modulations

4.75. We are now going to define a quantity called the "bandwidth" of a signal. Unfortunately, in practice, there isn't just one definition of bandwidth.

Definition 4.76. The **bandwidth** (BW) of a signal is usually calculated from the differences between two frequencies (called the bandwidth limits). Let's consider the following definitions of bandwidth for real-valued signals [3, p 173]

(a) **Absolute bandwidth**: Use the highest frequency and the lowest frequency in the positive-f part of the signal's nonzero magnitude spectrum.

- This uses the frequency range where 100% of the energy is confined.
- We can speak of absolute bandwidth if we have ideal filters and unlimited time signals.

(b) **3-dB bandwidth** (**half-power bandwidth**): Use the frequencies where the signal power starts to decrease by 3 dB (1/2).

- The magnitude is reduced by a factor of $1/\sqrt{2}$.

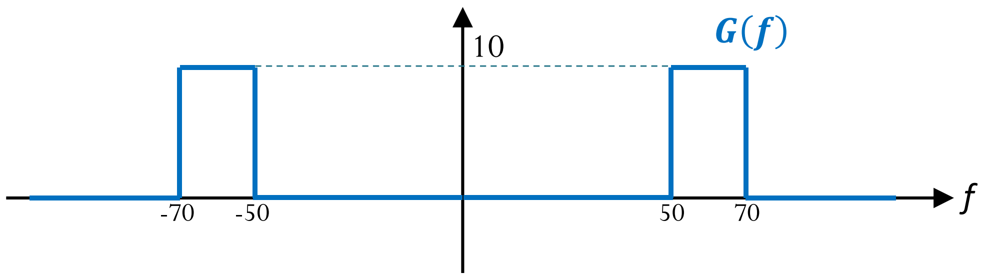
(c) **Null-to-null bandwidth**: Use the signal spectrum's first set of zero crossings.

(d) **Occupied bandwidth**: Consider the frequency range in which $X\%$ (for example, 99%) of the energy is contained in the signal's bandwidth.

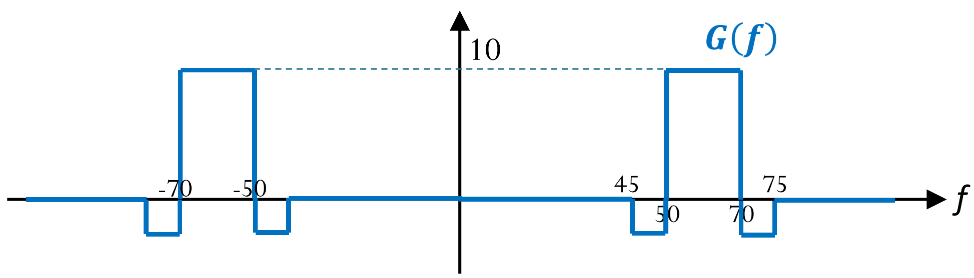
(e) **Relative power spectrum bandwidth**: the level of power outside the bandwidth limits is reduced to some value relative to its maximum level.

- Usually specified in negative decibels (dB).
- For example, consider a 200-kHz-BW broadcast signal with a maximum carrier power of 1000 watts and relative power spectrum bandwidth of -40 dB (i.e., 1/10,000). We would expect the station's power emission to not exceed 0.1 W outside of $f_c \pm 100$ kHz.

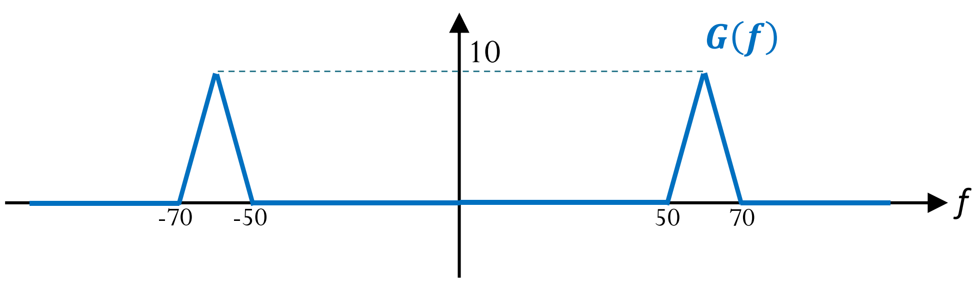
Example 4.77.



Example 4.78.



Example 4.79.



Example 4.80. Message bandwidth and the transmitted signal bandwidth

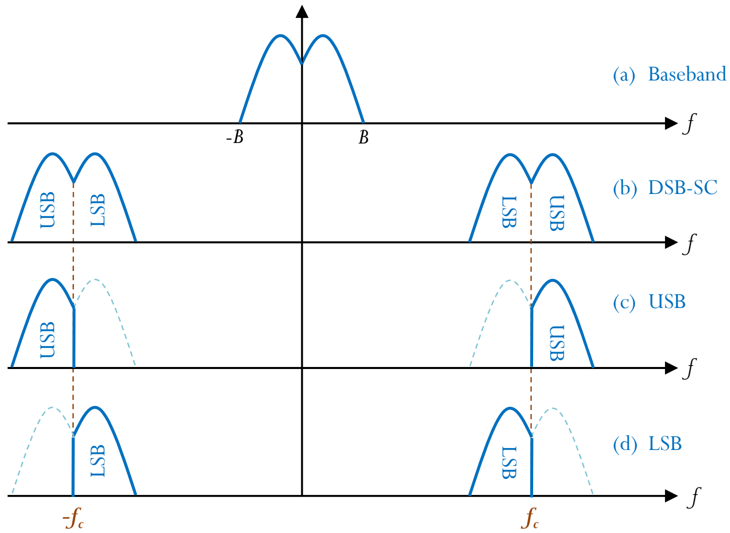


Figure 31: SSB spectra from suppressing one DSB sideband.

4.81. BW Inefficiency in DSB-SC system: Recall that for real-valued baseband signal $m(t)$, the conjugate symmetry property from 2.30 says that

$$M(-f) = (M(f))^* .$$

The DSB spectrum has two sidebands: the upper sideband (USB) and the lower sideband (LSB), each containing complete information about the baseband signal $m(t)$. As a result, DSB signals occupy twice the bandwidth required for the baseband.

4.82. Rough Approximation: If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz.

This result follows from the application of the width property[18] of convolution[19] to the convolution-in-frequency property.

Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz. We mentioned this property in 2.42.

[18]This property states that the width of $x * y$ is the sum of the widths of x and y.
[19]The width property of convolution does not hold in some pathological cases. See [5, p 98].

4.83. To improve the spectral efficiency of amplitude modulation, there exist two basic schemes to either utilize or remove the spectral redundancy:

(a) Single-sideband (SSB) modulation, which removes either the LSB or the USB so that for one message signal $m(t)$, there is only a bandwidth of B Hz.

(b) Quadrature amplitude modulation (QAM), which utilizes spectral redundancy by sending two messages over the same bandwidth of $2B$ Hz.

4.7 Single-Sideband Modulation

4.84. Transmitting both upper and lower sidebands of DSB is redundant. Transmission bandwidth can be cut in half if one sideband is suppressed along with the carrier.

Definition 4.85. Conceptually, in **single-sideband (SSB) modulation**, a sideband filter suppresses one sideband before transmission. [3, p 185–186]

(a) If the filter removes the lower sideband, the output spectrum consists of the upper sideband (USB) alone. Mathematically, the time domain representation of this SSB signal is

$$x_{\mathrm{USB}}(t) = m(t)\sqrt{2}\cos(2\pi f_c t) - m_h(t)\sqrt{2}\sin(2\pi f_c t). \qquad (62)$$

where $m_h(t)$ is the **Hilbert transform** of the message:

$$m_h(t) = \mathcal{H}\left\{m(t)\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m(\tau)}{t-\tau}d\tau = m(t) * \frac{1}{\pi t}. \qquad (63)$$

(b) If the filter removes the upper sideband, the output spectrum consists of the lower sideband (LSB) alone. Mathematically, the time domain representation of this SSB signal is

$$x_{\mathrm{LSB}}(t) = m(t)\sqrt{2}\cos(2\pi f_c t) + m_h(t)\sqrt{2}\sin(2\pi f_c t). \qquad (64)$$

Derivation of the time-domain representation is given in Section 4.9. More discussion on SSB can be found in [3, Sec 4.4], [14, Section 3.1.3] and [5, Section 4.5].

4.8 Quadrature Amplitude Modulation (QAM)

Definition 4.86. In ***quadrature amplitude modulation*** (***QAM***) or ***quadrature multiplexing***, two baseband real-valued signals $m_1(t)$ and $m_2(t)$ are transmitted simultaneously via the corresponding QAM signal:

$$x_{\mathrm{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t).$$

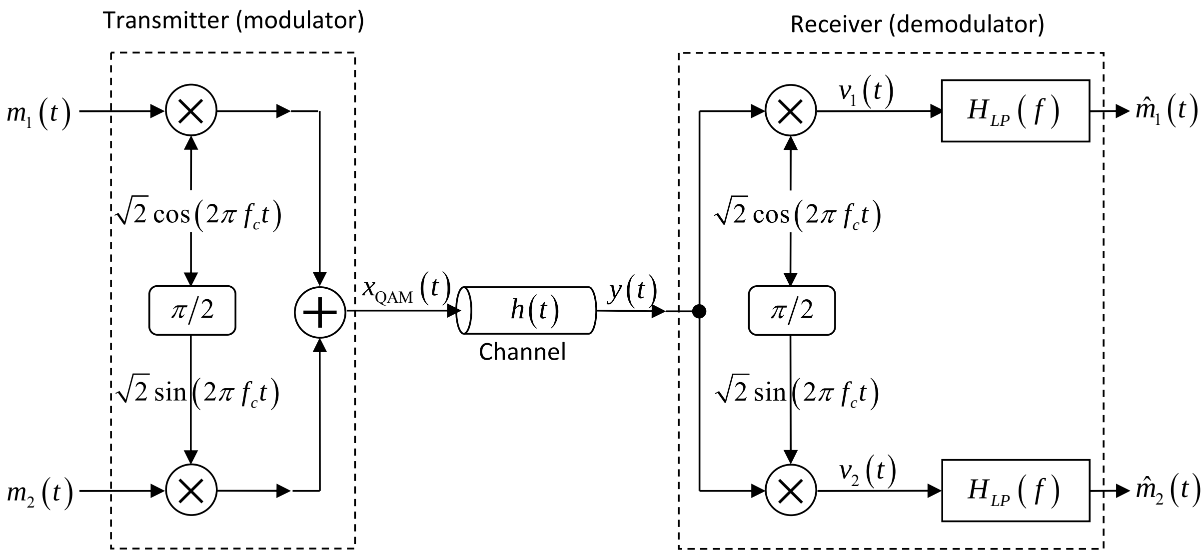


Figure 32: QAM Scheme

- QAM operates by transmitting two DSB signals via carriers of the same frequency but in phase quadrature.
- Both modulated signals simultaneously occupy the same frequency band.
- The "cos" (upper) channel is also known as the ***in-phase*** (***I***) channel and the "sin" (lower) channel is the ***quadrature*** (***Q***) channel.

4.87. ***Demodulation***: Under the usual assumption ($B < f_c$), the two baseband signals can be separated at the receiver by synchronous detection:

$$\mathrm{LPF}\left\{x_{\mathrm{QAM}}(t)\sqrt{2}\cos(2\pi f_c t)\right\} = m_1(t) \tag{65}$$

$$\mathrm{LPF}\left\{x_{\mathrm{QAM}}(t)\sqrt{2}\sin(2\pi f_c t)\right\} = m_2(t) \tag{66}$$

To see (65), note that

$$\begin{aligned}
v_1(t) &= x_{\text{QAM}}(t)\sqrt{2}\cos(2\pi f_c t) \\
&= \left(m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)\right)\sqrt{2}\cos(2\pi f_c t) \\
&= m_1(t)\,2\cos^2(2\pi f_c t) + m_2(t)\,2\sin(2\pi f_c t)\cos(2\pi f_c t) \\
&= m_1(t)\left(1+\cos(2\pi(2f_c)t)\right) + m_2(t)\sin(2\pi(2f_c)t) \\
&= m_1(t) + m_1(t)\cos(2\pi(2f_c)t) + m_2(t)\cos(2\pi(2f_c)t - 90^\circ)
\end{aligned}$$

- Observe that $m_1(t)$ and $m_2(t)$ can be separately demodulated.

Example 4.88. $(1)\sqrt{2}\cos(2\pi f_c t) + (1)\sqrt{2}\sin(2\pi f_c t)$

Example 4.89. $3\sqrt{2}\cos(2\pi f_c t) + 4\sqrt{2}\sin(2\pi f_c t)$

4.90. Suppose, during a time interval, the messages $m_1(t)$ and $m_2(t)$ are constant. Consider the signal $m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\sin(2\pi f_c t)$

4.91. Sinusoidal form (envelope-and-phase description [3, p. 165]):

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \phi(t)),$$

where

$$\begin{aligned}
\text{envelope:}\quad & E(t) = |m_1(t) - jm_2(t)| = \sqrt{m_1^2(t) + m_2^2(t)} \\
\text{phase:}\quad & \phi(t) = \angle\left(m_1(t) - jm_2(t)\right)
\end{aligned}$$

Example 4.92. In a QAM system, the transmitted signal is of the form

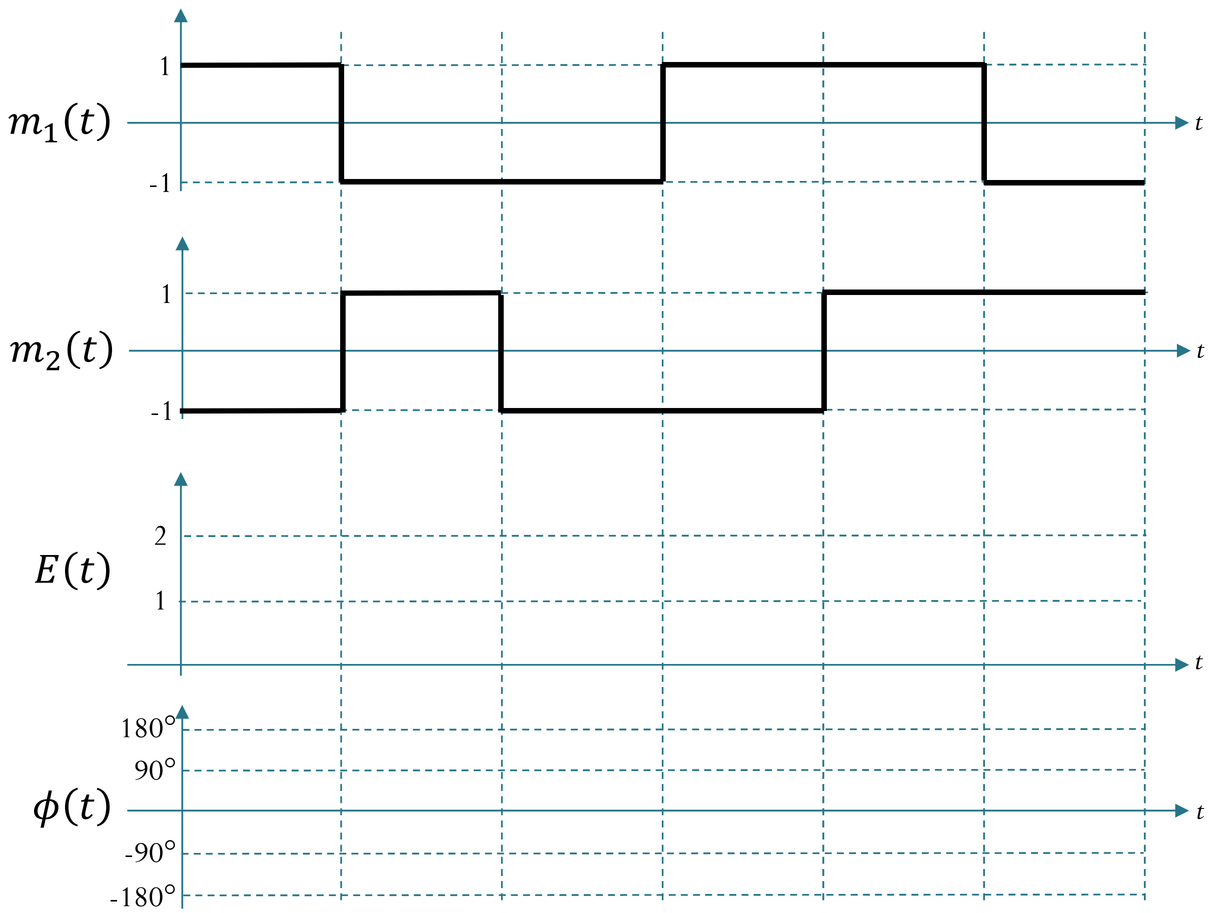
$$x_{\text{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t).$$

Here, we want to express $x_{\text{QAM}}(t)$ in the form

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \phi(t)),$$

where $E(t) \geq 0$ and $\phi(t) \in (-180°, 180°]$.

Consider $m_1(t)$ and $m_2(t)$ plotted in the figure below. Draw the corresponding $E(t)$ and $\phi(t)$.



4.93. $m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\sin(2\pi f_c t)$

4.94. Complex form:

$$x_{\text{QAM}}(t) = \sqrt{2}\text{Re}\left\{(m(t))\, e^{j2\pi f_c t}\right\}$$

where[20] $m(t) = m_1(t) - jm_2(t)$.

- We refer to $m(t)$ as the ***complex envelope*** (or ***complex baseband signal***) and the signals $m_1(t)$ and $m_2(t)$ are known as the ***in-phase*** and ***quadrature(-phase)*** components of $x_{\text{QAM}}(t)$.
- The term "quadrature component" refers to the fact that it is in phase quadrature ($\pi/2$ out of phase) with respect to the in-phase component.
- Key equation:

$$\text{LPF}\left\{\underbrace{\left(\text{Re}\left\{m(t) \times \sqrt{2}e^{j2\pi f_c t}\right\}\right)}_{x_{\text{QAM}}(t)} \times \left(\sqrt{2}e^{-j2\pi f_c t}\right)\right\} = m(t)\,.$$

4.95. Three equivalent ways of saying exactly the same thing:

(a) the complex-valued envelope $m(t)$ complex-modulates the complex carrier $e^{j2\pi f_c t}$,

- So, now you can understand what we mean when we say that a complex-valued signal is transmitted.

(b) the real-valued amplitude $E(t)$ and phase $\phi(t)$ real-modulate the amplitude and phase of the real carrier $\cos(2\pi f_c t)$,

(c) the in-phase signal $m_1(t)$ and quadrature signal $m_2(t)$ real-modulate the real in-phase carrier $\cos(2\pi f_c t)$ and the real quadrature carrier $\sin(2\pi f_c t)$.

[20] If we use $-\sin(2\pi f_c t)$ instead of $\sin(2\pi f_c t)$ for $m_2(t)$ to modulate,

$$\begin{aligned} x_{\text{QAM}}(t) &= m_1(t)\sqrt{2}\cos(2\pi f_c t) - m_2(t)\sqrt{2}\sin(2\pi f_c t) \\ &= \sqrt{2}\,\text{Re}\left\{m(t)\, e^{j2\pi f_c t}\right\} \end{aligned}$$

where

$$m(t) = m_1(t) + jm_2(t).$$

4.96. References: [3, p 164–166, 302–303], [14, Sect. 2.9.4], [5, Sect. 4.4], and [9, Sect. 1.4.1]

4.97. Question: In engineering and applied science, measured signals are real. Why should real measurable effects be represented by complex signals?

Answer: One complex signal (or channel) can carry information about two real signals (or two real channels), and the algebra and geometry of analyzing these two real signals as if they were one complex signal brings economies and insights that would not otherwise emerge. [9, p. 3]

4.9 More on Suppressed-Sideband Amplitude Modulation

4.98. There are a couple of important Fourier transform pairs[21] that haven't been discussed earlier.

(a) For the signum function,

$$\operatorname{sgn}(t) = \left\{ \begin{array}{ll} 1, & t > 0 \\ -1, & t < 0 \end{array} \right\} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{j\pi f} \tag{67}$$

To remember this, simply note that $\frac{d}{dt}\operatorname{sgn}(t) = 2\delta(t)$. Therefore,

$$\mathcal{F}\left\{\frac{d}{dt}\operatorname{sgn}(t)\right\} = \mathcal{F}\{2\delta(t)\} \equiv 2. \tag{68}$$

From the time differentiation property, we also have

$$\mathcal{F}\left\{\frac{d}{dt}\operatorname{sgn}(t)\right\} = j2\pi f \mathcal{F}\{\operatorname{sgn}(t)\} \tag{69}$$

Equating (68) and (69), we get (67). Note that such method is deceptively simple but does not highlight the difficulties inherent in the functions involved.

(b) For the unit-step function, because $u(t) = \frac{1+\operatorname{sgn}(t)}{2}$, we have

$$u(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.$$

(c) Applying the duality theorem to (67), we get

$$\frac{1}{\pi t} = h(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} H(f) = -j\operatorname{sgn}(f) = \left\{ \begin{array}{ll} -j, & f > 0 \\ j, & f < 0 \end{array} \right. = \left\{ \begin{array}{ll} 1 \cdot e^{-j\frac{\pi}{2}}, & f > 0 \\ 1 \cdot e^{j\frac{\pi}{2}}, & f < 0 \end{array} \right.$$

[21]Derivation of these pairs are not straight-forward. For those who are interested, please see B.L. Burrows and D.J. Colwell (1990): The Fourier transform of the unit step function, International Journal of Mathematical Education in Science and Technology, 21:4, 629–635

4.99. Let's define the right half and left half of $M(f)$ as $M_+(f)$ and $M_-(f)$, respectively. Observe that

$$M_+(f) \equiv \left\{ \begin{array}{ll} M(f), & f > 0 \\ 0, & f < 0 \end{array} \right\} = M(f)\,u(f) = M(f)\,\frac{1}{2}\,(1 + \operatorname{sgn}(f)) \tag{70}$$

From (63), applying the convolution-in-time property to the Hilbert transform of the message, we have

$$m(t) * \frac{1}{\pi t} = m_h(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M_h(f) = M(f) \times (-j\,\operatorname{sgn}(f)) \,. \tag{71}$$

Replacing $M(f)\,\operatorname{sgn}(f)$ in (70) with $jM_h(f)$, we have

$$M_+(f) = \frac{1}{2}\,(M(f) + jM_h(f)) \,.$$

Similarly,

$$M_-(f) \equiv \left\{ \begin{array}{ll} 0, & f > 0 \\ M(f), & f < 0 \end{array} \right\} = M(f)\,u(-f) = M(f)\,\frac{1}{2}\,(1 - \operatorname{sgn}(f)) = \frac{1}{2}\,(M(f) - jM_h(f)) \,.$$

Now, by the frequency-domain construction (in Figure 31c),

$$\begin{aligned} X_{\text{USB}}(f) &= AM_+(f - f_c) + AM_-(f - (-f_c)) \,, \\ &= \frac{A}{2}\,(M(f - f_c) + jM_h(f - f_c)) + \frac{A}{2}\,(M(f + f_c) - jM_h(f + f_c)) \,. \\ &= \frac{A}{2}\,(M(f - f_c) + M(f + f_c)) - \frac{A}{2j}\,(M_h(f - f_c) - M_h(f + f_c)) \,, \end{aligned}$$

With $A = \sqrt{2}$, the inverse Fourier transform is (62).

4.100. An SSB signal can be synchronously (coherently) demodulated just like DSB-SC signals. For example, multiplication of a USB signal by $\sqrt{2}\cos(2\pi f_c t)$ shifts its spectrum to the left and right by f_c, creating $M(f)$ around $f = 0$. Low-pass filtering of this signal yields the desired baseband signal. The case is similar with LSB signals.

Mathematically,

$$\begin{aligned} x_{\text{SSB}}(t)\,\sqrt{2}\cos(2\pi f_c t) &= \Big(m(t)\sqrt{2}\cos(2\pi f_c t) \mp m_h(t)\sqrt{2}\sin(2\pi f_c t)\Big)\,\sqrt{2}\cos(2\pi f_c t) \\ &= m(t)\,(1 + \cos(2\pi\,(2f_c)\,t)) \mp m_h(t)\sin(2\pi\,(2f_c)\,t) \\ &= m(t) + m(t)\cos(2\pi\,(2f_c)\,t) \mp m_h(t)\sin(2\pi\,(2f_c)\,t) \end{aligned}$$

Observe that

(a) If $m(t)$ is band-limited to B, then $m_h(t)$ is also band-limited to B because, from (71), we know that $M_h(f) = M(f) \times (-j\,\operatorname{sgn}(f))$. Therefore, the LPF that eliminates $m(t)\cos(2\pi\,(2f_c)\,t)$ will also eliminate $m_h(t)\sin(2\pi\,(2f_c)\,t)$.

(b) The product $x_{\text{SSB}}(t)\,\sqrt{2}\cos(2\pi f_c t)$ yields the baseband signal and another SSB signal with twice the carrier frequency.

4.101. An ideal Hilbert transformer (Hilbert phase shifter) is unrealizable (or realizable only approximately). This is due to an abrupt phase change of π at zero frequency.

Practical approximation of this ideal phase shifter still works fine when the message $m(t)$ has a dc null and very little low-frequency content.

Definition 4.102. In **vestigial-sideband modulation** (**VSB**) (or asymmetric sideband [6]), one sideband is passed almost completely while just a trace, or vestige, of the other sideband is included. [3, p 191–192]

4.103. In (analog) television video transmission, an AM wave is applied to a vestigial sideband filter. This modulation scheme is called **VSB plus carrier** (**VSB + C**). [3, p 193]

- The unsuppressed carrier allows for envelope detection, as in AM
  - Distortionless envelope modulation actually requires symmetric sidebands, but VSB + C can deliver a fair approximation.